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DISCONJUGACY AND TRANSFORMATIONS FOR SYMPLECTIC SYSTEMS

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ABSTRACT. We examine transformations and disconjugacy for general symplectic systems which include as special cases linear Hamiltonian difference systems and Sturm-Liouville difference equations of higher order. We give a Reid roundabout theorem for these systems and also for reciprocal symplectic systems. Particularly, we investigate a connection between eventual disconjugacy of linear Hamiltonian difference systems and their reciprocals. Finally, we present a disconjugacy-preserving transformation of a Sturm-Liouville equation of higher order which transforms this equation into another one of the same order.

1. Introduction. It has taken considerable effort to define disconjugacy for Sturm-Liouville difference equations of higher order

$$\text{(SL)} \qquad \sum_{\nu=0}^{n}(-1)^\nu \Delta^\nu\{r_k^{(\nu)}\Delta^\nu y_{k+n-\nu}\} = 0, \quad 0 \le k \le N$$

and to prove a so-called Reid roundabout theorem which contains the statement that disconjugacy is equivalent to positive definiteness of a certain related discrete quadratic functional. Recently, this problem was solved in [**10**] by treating (SL) as a special case of a linear Hamiltonian difference system

$$\text{(H)} \qquad \begin{aligned} \Delta x_k &= A_k x_{k+1} + B_k u_k, \\ \Delta u_k &= -C_k x_{k+1} - A_k^T u_k, \\ &\quad 0 \le k \le N \end{aligned}$$

(A, B, and C being square matrices) and by proving a Reid roundabout theorem for such more general systems.

In this paper we present an extension of those results to symplectic systems

$$\text{(S)} \qquad z_{k+1} = S_k z_k, \quad 0 \le k \le N$$

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where the matrices S_k are assumed to be symplectic. We define disconjugacy for symplectic systems and give conditions which are equivalent to the positive definiteness of the discrete quadratic functional

$$\mathcal{F}(z) = \sum_{k=0}^{N} z_k^T \{S_k^T \mathcal{K} S_k - \mathcal{K}\} z_k.$$

For a more precise discussion of the above properties and quantities, e.g., $r_k^{(\nu)}$, A_k, B_k, C_k, x_k, u_k, S_k, z_k, $\mathcal{K}$, we refer to the next section.

The following are some of the advantages of our (symplectic) approach. First of all, any system (H) is covered by a system (S) but not the other way around. So this gives access to many more systems and at the same time to many more discrete quadratic functionals which may arise as a second variation when trying to solve variational problems in control theory. Secondly, the objects connected to the system (S), e.g., the so-called Riccati operator which plays an important role for the characterization of positive definiteness, the results and their proofs, e.g., results on certain transformations of a symplectic system into another symplectic system, read much smoother and easier compared to those for systems (H), although systems (S) are more general. This is basically a consequence of the fact that all proofs for (H) need in the essence its symplectic structure only, but this symplectic structure is very well hidden in systems (H). Finally, by looking at reciprocal symplectic systems

$$(\tilde{\mathrm{S}}) \qquad z_k = \tilde{S}_k z_{k+1}, \quad 0 \le k \le N$$

it is possible to give results for reciprocal linear Hamiltonian difference systems that have not been obtained in the previous literature. Our main result concerning this topic states that a system (H) is under certain additional assumptions eventually disconjugate if and only if its reciprocal system is eventually disconjugate.

Let us briefly give an overview on the literature related to the above. While (SL) for $n \in \mathbf{N}$ has been investigated, e.g., by C. Ahlbrandt, P. Hartman, and A. Peterson (see [**7, 24**]), there is a long list of authors who dealt with (SL) in case $n = 1$, see for example [**12, 23, 25, 26, 34**]. This is essentially because the B from (H) which corresponds to (SL) for $n = 1$ is invertible which is the easier case when looking at systems

(H). The more general case of (H) with invertible B was first examined in a series of four papers by L. Erbe and P. Yan in [**19, 20, 21, 22**] and also by C. Ahlbrandt, S. Chen, O. Došlý, M. Heifetz, J. Hooker, T. Peil, A. Peterson, and J. Ridenhour (see [**2, 3, 4, 13, 29, 30, 31, 32, 33**]), while singular B has been allowed in the papers by M. Bohner [**9, 10**]. Finally, general symplectic systems were introduced by C. Ahlbrandt and A. Peterson in [**7**].

What follows is a short summary of the set up of this paper. The next section contains preliminary and partly well-known results on symplectic systems and the corresponding functionals. In Section 3 we introduce the concept of focal points for matrix-valued solutions of (S) as well as the concept of generalized zeros for vector-valued solutions of (S). Disconjugacy is defined in terms of generalized zeros and a Reid roundabout theorem is proved. This theorem may be viewed as a discrete analogue of W.T. Reid's original (continuous) theorem (see [**36**, Chapter VII], [**37**, Theorem V.6.3] and [**27**, Theorem 2.4.1]). Moreover, we discuss the above concepts for reciprocal systems ($\tilde{\mathrm{S}}$) and prove a Reid roundabout theorem for those systems also. In this section we also present transformations of symplectic systems which preserve the important property of disconjugacy. With the aid of the results of Section 3 we investigate in Section 4 a connection of systems (H) with their corresponding reciprocal systems. Finally, we obtain in Section 5 a result on disconjugacy-preserving transformations for Sturm-Liouville difference equations. This result may be viewed as the discrete counterpart of the (continuous) transformation method suggested by C. Ahlbrandt, D. Hinton and R. Lewis in [**6**], and it complements the results on transformations for systems (H) obtained by O. Došlý in [**18**].

2. Preliminary definitions and results. Let $n, N \in \mathbf{N}$, $J := [0, N] \cap \mathbf{Z}$, $J^* := [0, N+1] \cap \mathbf{Z}$. For a matrix- or vector-valued function f defined on a subset of J^* we write $f_k := f(k)$, and we sometimes refer to the matrix or vector f using a slight abuse of language. The difference operator Δ is defined by $\Delta f_k := f_{k+1} - f_k$ while the shift operator E is given by $Ef_k := f_{k+1}$. By $M^\dagger$ we denote the Moore-Penrose inverse of a matrix M, i.e., the unique matrix satisfying $MM^\dagger M = M$ and $M^\dagger M M^\dagger = M^\dagger$ such that both $MM^\dagger$ and $M^\dagger M$ are symmetric, see for example [**8**]. For a symmetric matrix D we write $D > 0$ if D is positive definite and $D \geq 0$ if D is positive semidefinite. By $\operatorname{Ker} M$, $\operatorname{Im} M$, $\operatorname{rank} M$, M^T, and M^{-1} we denote the kernel, image, rank, transpose, and inverse of the matrix M, and we abbreviate $(M^T)^{-1}$ by M^{T-1}. We use I for the $n \times n$-identity matrix, and we also need the $2n \times 2n$-matrices

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{K} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

Finally, a $2n \times 2n$-matrix S is called *symplectic* if $S^T \mathcal{J} S = \mathcal{J}$ holds.

Definition 1. Let S be symplectic and $z : J^* \to \mathbf{R}^{2n}$.

(i) z satisfies the *boundary conditions* if $\mathcal{K} z_0 = \mathcal{K} z_{N+1} = 0$ and it is *admissible* if it satisfies the *equation of motion* $\mathcal{K} S z = \mathcal{K} E z$ on J. The *Euler-Lagrange equation* is given by $\mathcal{K}^T S z = \mathcal{K}^T E z$ on J and the corresponding *symplectic system* (S) by $Sz = Ez$ on J.

(ii) The *discrete quadratic functional* $\mathcal{F}(z) = \sum_{k=0}^{N} z_k^T \{S_k^T \mathcal{K} S_k - \mathcal{K}\} z_k$ is called *positive definite* (we write $\mathcal{F} > 0$) if $\mathcal{F}(z) > 0$ for all admissible z satisfying the boundary conditions with $\mathcal{K} z \neq 0$ on J.

Remark 1. (i) We have $\mathcal{J}^{-1} = \mathcal{J}^T = -\mathcal{J}$ and $\mathcal{J} = \mathcal{K}^T - \mathcal{K}$. If S is symplectic, then S is invertible and $S^{-1} = \mathcal{J} S^T \mathcal{J}^T$, $S^T = \mathcal{J} S^{-1} \mathcal{J}^T$ hold. Furthermore, S^{-1} and S^T are then symplectic also.

(ii) For a symplectic system *Wronski's identity* holds, i.e., if vectors or matrices f and g solve (S), then $f_k^T \mathcal{J} g_k$ is constant on J. To see this, just note that, due to simplecticity,

$$f_{k+1}^T \mathcal{J} g_{k+1} = f_k^T S_k^T \mathcal{J} S_k g_k = f_k^T \mathcal{J} g_k$$

holds. We denote vector-valued solutions $z : J^* \to \mathbf{R}^{2n}$ of (S) by small letters and use capital letters for $2n \times n$-matrix-valued solutions Z of (S). Such a Z is called a *conjoined basis* of (S) if $Z^T \mathcal{J} Z = 0$ and $\operatorname{rank} Z = n$ hold on J^*. Two conjoined bases Z and $\tilde{Z}$ are called *normalized* whenever $Z^T \mathcal{J} \tilde{Z} = I$ holds, and this is true if and only if the matrix $\begin{pmatrix} Z & \tilde{Z} \end{pmatrix}$ is symplectic. Note also that, for any conjoined basis Z, we can find another conjoined basis $\tilde{Z}$ such that Z and $\tilde{Z}$ are normalized; just choose $\tilde{Z}$ to be the (unique) solution of (S) satisfying $\tilde{Z}_0 = \mathcal{J}^{-1} Z_0 (Z_0^T Z_0)^{-1}$. Finally, the solution Z of (S) with $Z_m = \begin{pmatrix} 0 \\ I \end{pmatrix}$

is called the *principal solution* of (S) at m while the solution $\tilde{Z}$ of (S) with $\tilde{Z}_m = \binom{-I}{0}$ is called the *associated solution* of (S) at m.

(iii) Whenever we do not make any other assumptions, we will put

$$S = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} x \\ u \end{pmatrix}$$

with $x, u : J^* \to \mathbf{R}^n$ and $n \times n$-matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. Then z is admissible if $x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k$ holds, and it satisfies the Euler-Lagrange equation in case of $u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k$ for all $k \in J$. Also, the conditions

$$\begin{cases} \mathcal{A}^T\mathcal{D} - \mathcal{C}^T\mathcal{B} = \mathcal{A}\mathcal{D}^T - \mathcal{B}\mathcal{C}^T = \mathcal{D}^T\mathcal{A} - \mathcal{B}^T\mathcal{C} = \mathcal{D}\mathcal{A}^T - \mathcal{C}\mathcal{B}^T = I; \\ \mathcal{A}\mathcal{B}^T, \mathcal{C}\mathcal{D}^T, \mathcal{C}^T\mathcal{A}, \mathcal{D}^T\mathcal{B}, \mathcal{B}\mathcal{A}^T, \mathcal{D}\mathcal{C}^T, \mathcal{A}^T\mathcal{C}, \mathcal{B}^T\mathcal{D} \text{ symmetric} \end{cases}$$

are then easily seen to be necessary and sufficient for S to be symplectic. In this case, one may easily check the formula

$$S^T\mathcal{K}S - \mathcal{K} = \begin{pmatrix} \mathcal{C}^T\mathcal{A} & \mathcal{C}^T\mathcal{B} \\ \mathcal{B}^T\mathcal{C} & \mathcal{D}^T\mathcal{B} \end{pmatrix},$$

so that $\mathcal{F}$ now reads

$$\mathcal{F}(x,u) = \sum_{k=0}^{N} \{x_k^T \mathcal{C}_k^T \mathcal{A}_k x_k + 2x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k\}.$$

For admissible (x, u) it is readily verified that this formula reduces to

$$\mathcal{F}(x,u) = x_{N+1}^T u_{N+1} - x_0^T u_0 + \sum_{k=0}^{N} x_{k+1}^T \{\mathcal{C}_k x_k + \mathcal{D}_k u_k - u_{k+1}\},$$

and this formula especially helps in those cases when $Eu = \mathcal{C}x + \mathcal{D}u$ holds on a subset of J.

(iv) For $k \in J^* \setminus \{0\}$ we define controllability matrices by

$$G_k = (\mathcal{A}_{k-1}\mathcal{A}_{k-2}\cdots\mathcal{A}_1\mathcal{B}_0 \quad \mathcal{A}_{k-1}\mathcal{A}_{k-2}\cdots\mathcal{A}_2\mathcal{B}_1 \quad \ldots \quad \mathcal{A}_{k-1}\mathcal{B}_{k-2} \quad \mathcal{B}_{k-1}).$$

Then it is easy to check by induction that (x, u) is admissible with $x_0 = 0$ if and only if

$$x_k = G_k \begin{pmatrix} u_0 \\ \vdots \\ u_{k-1} \end{pmatrix}$$

holds for all $k \in J^* \setminus \{0\}$.

(v) Let (X, U) be a conjoined basis of (S) with $\operatorname{Ker} EX \subset \operatorname{Ker} X$ on J. Then we have $\operatorname{Ker} EX^T \subset \operatorname{Ker} \mathcal{B}^T$ and $x \in \operatorname{Im} X$ on J for any admissible (x, u) with $x_0 = 0$. To prove the first claim pick a conjoined basis $(\tilde{X}, \tilde{U})$ such that (X, U) and $(\tilde{X}, \tilde{U})$ are normalized, compare (iii), and $c \in \operatorname{Ker} X_{k+1}^T$, so that $X_{k+1}\tilde{X}_{k+1}^T c = \tilde{X}_{k+1} X_{k+1}^T c = 0$ and

$$\begin{aligned} \mathcal{B}_k^T c &= \tilde{X}_k X_{k+1}^T c + \mathcal{B}_k^T c \\ &= \tilde{X}_k X_k^T \mathcal{A}_k^T c + (I + \tilde{X}_k U_k^T)\mathcal{B}_k^T c \\ &= X_k \tilde{X}_{k+1}^T c = 0 \end{aligned}$$

hold. For the second claim, we note that $x_0 \in \operatorname{Im} X_0$ is surely true and that $x_k = X_k c \in \operatorname{Im} X_k$ implies

$$x_{k+1} = \mathcal{A}_k X_k c + \mathcal{B}_k u_k = X_{k+1} c + \mathcal{B}_k(u_k - U_k c) \in \operatorname{Im} X_{k+1}.$$

Finally note that $\operatorname{Ker} V \subset \operatorname{Ker} W$ if and only if $W = WV^\dagger V$, see [**9**, Lemma 4].

Definition 2. (i) We say that a scalar function $y : [0, N + 2n] \cap \mathbf{Z} \to \mathbf{R}$ satisfies the *Sturm-Liouville difference equation* (SL) given by reals $r_k^{(\nu)}$, $k \in J$, $\nu \in [0, n] \cap \mathbf{Z}$ with $r_k^{(n)} \neq 0$ for $k \in J$ if $\sum_{\nu=0}^{n} (-\Delta)^\nu \{r_k^{(\nu)} \Delta^\nu y_{k+n-\nu}\} = 0$ holds on J.

(ii) $\binom{x}{u} : J \to \mathbf{R}^{2n}$ solves the *linear Hamiltonian difference system* (H) given by the symmetric $n \times n$-matrices B, C, and by the $n \times n$-matrix A such that $\tilde{A} := (I - A)^{-1}$ exists if both $\Delta x_k = A_k x_{k+1} + B_k u_k$ and $\Delta u_k = -C_k x_{k+1} - A_k^T u_k$ hold on J.

Remark 2. (i) As is well known, see, e.g., [**22**, Section 3] or [**9**, Lemma 2], any equation (SL) is equivalent to a system (H) with the

$n \times n$-matrices

$$A_k \equiv (a_{ij}), \quad B_k = \operatorname{diag}\left(0, \dots, 0, \frac{1}{r_k^{(n)}}\right),$$
$$-C_k = \operatorname{diag}(r_k^{(0)}, r_k^{(1)}, \dots, r_k^{(n-1)})$$

(where $a_{ij} = 1$ if $j = i + 1$ and 0 otherwise) in the following sense: (x, u) solves (H) if and only if y solves (SL) with

$$\begin{bmatrix} x_k^{(\nu)} = \Delta^{\nu-1} y_{k+n-\nu}, \\ u_k^{(\nu)} = \sum_{\mu=\nu}^{n} (-\Delta)^{\mu-\nu} \{r_k^{(\mu)} \Delta^\mu y_{k+n-\mu}\} \end{bmatrix}, \quad 1 \le \nu \le n, \quad k \in J.$$

(ii) Invertibility of $I - A$ ensures that (H) may be equivalently written as

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = S_k^{(H)} \begin{pmatrix} x_k \\ u_k \end{pmatrix}$$

with

$$S^{(H)} = \begin{pmatrix} \tilde{A} & \tilde{A}B \\ -C\tilde{A} & -C\tilde{A}B + I - A^T \end{pmatrix}.$$

Note that $S^{(H)}$ is symplectic so that any (H) is a symplectic system. Furthermore the corresponding functional then may be computed to be, compare Remark 1 (iii),

$$\mathcal{F}(x, u) = \sum_{k=0}^{N} \{u_k^T B_k u_k - x_{k+1}^T C_k x_{k+1}\}$$

whenever (x, u) is admissible, i.e., whenever $x_{k+1} = \tilde{A}_k x_k + \tilde{A}_k B_k u_k$ or equivalently $\Delta x_k = A_k x_{k+1} + B_k u_k$ holds.

Lemma 1. *The symplectic system $z_{k+1} = S_k z_k$ with*

$$S = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$$

is a system of the form (H) *if and only if the matrix $\mathcal{A}$ is invertible.*

Proof. One direction of the claim is already clear because of the preceding Remark 2 (ii). Now define $A := I - \mathcal{A}^{-1}$, $B := \mathcal{A}^{-1}\mathcal{B}$, $C := -\mathcal{C}\mathcal{A}^{-1}$, and then symplecticity forces, see Remark 1 (iii), B and C to be symmetric and

$$\begin{aligned}\mathcal{D} &= \mathcal{A}^{T-1}(I + \mathcal{C}^T\mathcal{B})\\ &= \mathcal{A}^{T-1} + \mathcal{C}\mathcal{A}^{-1}\mathcal{B}\\ &= I - A^T - C(I-A)^{-1}B\end{aligned}$$

so that (S) may be written as a Hamiltonian system. $\square$

3. Disconjugacy for symplectic systems and for reciprocal symplectic systems. Now we may proceed similarly as in [**10**] to prove our main result of this section, Theorem 1 below. We will need four lemmas for the proof of the Reid roundabout theorem for general symplectic systems, and for this purpose we put for convenience

$$z = \begin{pmatrix} x \\ u \end{pmatrix}, \qquad Z = \begin{pmatrix} X \\ U \end{pmatrix}, \qquad S = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix},$$

and for symmetric Q

$$\begin{cases} R_k[Q] = \begin{pmatrix} I \\ Q_{k+1} \end{pmatrix}^T \mathcal{J}^T S_k \begin{pmatrix} I \\ Q_k \end{pmatrix} \\ \qquad\quad = Q_{k+1}(\mathcal{A}_k + \mathcal{B}_k Q_k) - (\mathcal{C}_k + \mathcal{D}_k Q_k), \\ P_k[Q] = \begin{pmatrix} 0 \\ I \end{pmatrix}^T S_k^T \begin{pmatrix} I \\ 0 \end{pmatrix} \begin{pmatrix} I \\ Q_{k+1} \end{pmatrix}^T \mathcal{J} S_k \begin{pmatrix} 0 \\ I \end{pmatrix} = \mathcal{B}_k^T \mathcal{D}_k - \mathcal{B}_k^T Q_{k+1} \mathcal{B}_k. \end{cases}$$

Lemma 2. *For admissible* (x,u) *and symmetric* Q *we put* $s := u - Qx$. *Then*

(i)

$$\begin{aligned}&\Delta\{x_k^T Q_k x_k\} - x_k^T \mathcal{C}_k^T \mathcal{A}_k x_k - u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k - 2u_k^T \mathcal{B}_k^T \mathcal{C}_k x_k + s_k^T P_k[Q] s_k \\ &\qquad = 2u_k^T \mathcal{B}_k^T R_k[Q] x_k + x_k^T\{R_k^T[Q]\mathcal{A}_k - Q_k \mathcal{B}_k^T R_k[Q]\} x_k,\end{aligned}$$

(ii) $$\{\mathcal{D}_k^T - \mathcal{B}_k^T Q_{k+1}\} x_{k+1} = x_k + P_k[Q] s_k - \mathcal{B}_k^T R_k[Q] x_k.$$

Proof. Some computations (compare also [**10**, Lemma 2]), using admissibility of (x,u), i.e., $x_{k+1} = \mathcal{A}_k x_k + B_k u_k$, easily yield the above claims. $\square$

Lemma 3. *Let* (X,U) *be a conjoined basis of* (S) *with* $\operatorname{Ker} EX \subset \operatorname{Ker} X$ *on* J *and suppose that* Q *is symmetric with* $QX = UX^\dagger X$. *Then we have*

$$R_k[Q]X_k = 0 \quad \text{and} \quad P_k[Q] = X_k X_{k+1}^\dagger \mathcal{B}_k.$$

Proof. First, Remark 1 (v) yields $X_k X_{k+1}^\dagger X_{k+1} = X_k$ and $X_{k+1}X_{k+1}^\dagger \mathcal{B}_k = \mathcal{B}_k$. Now

$$\begin{aligned} R_k[Q]X_k &= \begin{pmatrix} I \\ Q_{k+1} \end{pmatrix}^T \mathcal{J}^T S_k \begin{pmatrix} X_k \\ U_k \end{pmatrix} X_k^\dagger X_k \\ &= \begin{pmatrix} Q_{k+1} \\ -I \end{pmatrix}^T \begin{pmatrix} X_{k+1} \\ U_{k+1} \end{pmatrix} X_k^\dagger X_k = 0 \end{aligned}$$

yields the first claim, while

$$P_k[Q] = \mathcal{D}_k^T X_{k+1} X_{k+1}^\dagger \mathcal{B}_k - \mathcal{B}_k^T Q_{k+1} X_{k+1} X_{k+1}^\dagger \mathcal{B}_k = X_k X_{k+1}^\dagger \mathcal{B}_k$$

takes care of the second statement. $\square$

Definition 3. A conjoined basis (X,U) of (S) has a *focal point* in $(k, k+1]$, $k \in J$, if

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k \quad \text{and} \quad X_k X_{k+1}^\dagger \mathcal{B}_k \geq 0$$

does not hold.

Lemma 4. *Suppose that for all solutions* (x,u) *of* (S) *with* $x_0 = 0$ *we have that* $x_m^T c > 0$ *whenever* $x_m \neq 0$ *and* $x_{m+1} = \mathcal{B}_m c$ *hold. Then the principal solution of* (S) *at* 0 *has no focal points in* $(0, N+1]$.

Proof. First, let $\alpha \in \operatorname{Ker} X_{m+1}$ and put

$$\begin{pmatrix} x \\ u \end{pmatrix} := \begin{pmatrix} X \\ U \end{pmatrix} \alpha.$$

Then (x,u) solves (S) with $x_0=0$, $x_{m+1}=0\in\operatorname{Im}\mathcal{B}_m$ so that $x_m=0$ also, i.e., $\operatorname{Ker}X_{m+1}\subset\operatorname{Ker}X_m$ holds. Now, let $c\in\mathbf{R}^n$, $\alpha:=X_{m+1}^\dagger\mathcal{B}_m c$, and $\binom{x}{u}:=\binom{X}{U}\alpha$. Again, (x,u) solves (S) with $x_0=0$, $x_{m+1}=\mathcal{B}_m c$, and $x_m^Tc=c^TX_mX_{m+1}^\dagger\mathcal{B}_m c$ so that $X_mX_{m+1}^\dagger\mathcal{B}_m\geq 0$ holds also. □

Definition 4. A (vector-valued) solution (x,u) of (S) has a *generalized zero* in $(k,k+1]$, $k\in J$, if

$$x_k\neq 0,\qquad x_{k+1}\in\operatorname{Im}\mathcal{B}_k,\quad\text{and}\quad x_k^T\mathcal{B}_k^\dagger x_{k+1}\leq 0$$

hold. (S) is called *disconjugate* on J if no solution of (S) has more than one and if no solution (x,u) of (S) with $x_0=0$ has any generalized zeros on J.

Remark 3. Suppose $x_{m+1}=\mathcal{B}_m c\in\operatorname{Im}\mathcal{B}_m$ for a solution (x,u) of (S). Then $x_m^T\mathcal{B}_m^\dagger x_{m+1}=x_m^Tc$. To see this, note that $x_{m+1}=\mathcal{B}_m c$ implies

$$x_m=\mathcal{D}_m^Tx_{m+1}-\mathcal{B}_m^Tu_{m+1}=\mathcal{B}_m^T(\mathcal{D}_mc-u_{m+1})$$

and $x_m^T\mathcal{B}_m^\dagger x_{m+1}=(\mathcal{D}_mc-u_{m+1})^T\mathcal{B}_m\mathcal{B}_m^\dagger\mathcal{B}_mc=x_m^Tc$.

Lemma 5. *If $\mathcal{F}>0$, then* (S) *is disconjugate on J.*

Proof. Suppose that (S) is not disconjugate, i.e., (see Remark 3) there exist $m,p\in J$ and $c_m,c_p\in\mathbf{R}^n$ with $m<p$ and

$$\begin{gathered}x_{m+1}=\mathcal{B}_mc_m,\qquad x_{p+1}=\mathcal{B}_pc_p,\\ x_p\neq 0,\qquad x_m^Tc_m\leq 0,\qquad x_p^Tc_p\leq 0,\end{gathered}$$

where (x,u) is some solution of (S). Define

$$\tilde{x}_k:=\begin{cases}x_k & m+1\leq k\leq p\\ 0 & \text{otherwise,}\end{cases}$$

$$\tilde{u}_k:=\begin{cases}c_m & k=m\\ u_k & m+1\leq k\leq p-1\\ u_p-c_p & k=p\\ 0 & \text{otherwise,}\end{cases}$$

so that $\tilde{x}_0 = \tilde{x}_{N+1} = 0$ and $\tilde{x}_p \neq 0$ hold. Moreover, $(\tilde{x}, \tilde{u})$ is easily checked to be admissible and $\mathcal{F}(\tilde{x}, \tilde{u}) = x_m^T c_m + x_p^T c_p \leq 0$ can be verified by applying the formula in Remark 1 (iii). Thus $\mathcal{F} \not> 0$. $\square$

Remark 4. We are now ready to prove our main result of this section, the Reid roundabout theorem for general symplectic systems

$$\text{(S)} \qquad z_{k+1} = S_k z_k, \quad k \in J.$$

Lemmas 2 through 5 will easily yield that (i), (ii), (iii), (iv), and (v) below are equivalent. Applying this result to the system

$$(\tilde{\text{S}}) \qquad z_k = S_k^{-1} z_{k+1}, \quad k \in J,$$

the equivalence of (vi), (vii), (viii), (ix), and (x) will follow similarly. Focal points of matrix-valued solutions and generalized zeros of vector-valued solutions of $(\tilde{\text{S}})$ (and thus disconjugacy of $(\tilde{\text{S}})$) are defined roughly speaking by interchanging the roles of k and $k+1$; for convenience, we repeat the precise definitions in the statements (vi) through (x) of the theorem below. Observe also that z (or Z) solves (S) if and only if z (or Z) solves $(\tilde{\text{S}})$. Thus, we should actually talk about focal points with respect to the reciprocal system $(\tilde{\text{S}})$ and also about generalized zeros with respect to the reciprocal system $(\tilde{\text{S}})$, but for convenience we will again use the terms focal point and generalized zero. Finally, as a third step of the proof, we will show the equivalence of statements (i) and (vi). Note that, in Theorem 1 below, we again let S be symplectic and put

$$\mathcal{K} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \qquad z = \begin{pmatrix} x \\ u \end{pmatrix},$$
$$Z = \begin{pmatrix} X \\ U \end{pmatrix}, \quad \text{and} \quad S = \begin{pmatrix} \mathcal{A} & B \\ \mathcal{C} & \mathcal{D} \end{pmatrix}.$$

Theorem 1 (Reid roundabout theorem for symplectic systems). *The following statements are equivalent.*

(i) $\mathcal{F}(z) = \sum_{k=0}^{N} z_k^T \{S_k^T \mathcal{K} S_k - \mathcal{K}\} z_k > 0$ *for all* z *with*

$$\mathcal{K}z \neq 0, \qquad \mathcal{K}z_0 = \mathcal{K}z_{N+1} = 0,$$

and

$$\mathcal{K}z_{k+1} = \mathcal{K}S_k z_k, \quad k \in J.$$

(ii) (S) *is disconjugate on* J; *i.e., no solution of* (S) *has more than one and no solution* z *of* (S) *with* $\mathcal{K}z_0 = 0$ *has any generalized zeros in* $(0, N+1]$, *where* $(m, m+1]$ *contains a generalized zero of a solution* z *of* (S) *if*

$$x_m \neq 0, \qquad x_{m+1} \in \operatorname{Im} \mathcal{B}_m,$$

and

$$x_m^T \mathcal{B}_m^\dagger x_{m+1} \leq 0 \quad \text{hold.}$$

(iii) *No solution* z *of* (S) *with* $\mathcal{K}z_0 = 0$ *has any generalized zero in* $(0, N+1]$.

(iv) *The solution* Z *of* (S) *with* $Z_0 = \begin{pmatrix} 0 \\ I \end{pmatrix}$ *has no focal points in* $(0, N+1]$*:*

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k \quad \textit{and} \quad X_k X_{k+1}^\dagger \mathcal{B}_k \geq 0 \quad \textit{hold for all } k \in J.$$

(v) $R_k[Q]G_k = 0$ *has a symmetric solution* Q *on* J *with* $P_k[Q] \geq 0$; *where*

$$\begin{aligned} R_k[Q] &= \begin{pmatrix} I \\ Q_{k+1} \end{pmatrix}^T \mathcal{J}^T S_k \begin{pmatrix} I \\ Q_k \end{pmatrix}, \\ P_k[Q] &= \mathcal{B}_k^T \mathcal{D}_k - \mathcal{B}_k^T Q_{k+1} \mathcal{B}_k, \end{aligned}$$

and

$$G_k = (\mathcal{A}_{k-1}\mathcal{A}_{k-2}\cdots\mathcal{A}_1\mathcal{B}_0 \quad \mathcal{A}_{k-1}\mathcal{A}_{k-2}\cdots\mathcal{A}_2\mathcal{B}_1 \quad \ldots \quad \mathcal{A}_{k-1}\mathcal{B}_{k-2} \quad \mathcal{B}_{k-1}).$$

(vi) $\tilde{\mathcal{F}}(z) := \sum_{k=0}^N z_{k+1}^T \{S_k^{T-1}\mathcal{K}S_k^{-1} - \mathcal{K}\} z_{k+1} < 0$ *for all* z *with*

$$\mathcal{K}z \neq 0, \qquad \mathcal{K}z_0 = \mathcal{K}z_{N+1} = 0, \quad \textit{and} \quad \mathcal{K}z_k = \mathcal{K}S_k^{-1} z_{k+1},\ k \in J.$$

(vii) $(\tilde{\mathrm{S}})$ *is disconjugate on* J*, i.e., no solution of* (S) *has more than one and no solution* z *of* (S) *with* $\mathcal{K}z_{N+1}=0$ *has any generalized zeros on* $[0,N+1)$*, where* $[m,m+1)$ *contains a generalized zero of a solution* z *of* (S) *if*

$$x_{m+1}\neq 0, \qquad x_m\in \operatorname{Im}\mathcal{B}_m^T, \quad \text{and} \quad x_m^T\mathcal{B}_m^\dagger x_{m+1}\leq 0 \quad \text{hold.}$$

(viii) *No solution* z *of* (S) *with* $\mathcal{K}z_{N+1}=0$ *has any generalized zero on* $[0,N+1)$.

(ix) *The solution* Z *of* (S) *with* $Z_{N+1}=\begin{pmatrix}0\\-I\end{pmatrix}$ *has no focal points in* $[0,N+1)$:

$$\operatorname{Ker}X_k\subset \operatorname{Ker}X_{k+1} \quad \text{and} \quad X_{k+1}X_k^\dagger\mathcal{B}_k^T\geq 0 \quad \text{hold for all } k\in J.$$

(x) $\tilde{R}_k[Q]\tilde{G}_k=0$ *has a symmetric solution* Q *on* J *with* $\tilde{P}_k[Q]\geq 0$; *where*

$$\tilde{R}_k[Q]=\begin{pmatrix}Q_k\\I\end{pmatrix}^T\mathcal{J}S_k^T\begin{pmatrix}Q_{k+1}\\I\end{pmatrix}, \qquad \tilde{P}_k[Q]=\mathcal{B}_k\mathcal{A}_k^T-\mathcal{B}_kQ_k\mathcal{B}_k^T,$$

and

$$\tilde{G}_k=(\mathcal{D}_{k+1}^T\mathcal{D}_{k+2}^T\cdots\mathcal{D}_{N-1}^T\mathcal{B}_N^T \quad \mathcal{D}_{k+1}^T\mathcal{D}_{k+2}^T\cdots\mathcal{D}_{N-2}^T\mathcal{B}_{N-1}^T \quad \cdots \quad \mathcal{D}_{k+1}^T\mathcal{B}_{k+2}^T \quad \mathcal{B}_{k+1}^T).$$

Proof. (i) implies (ii) by Lemma 5, (iii) follows from (ii) trivially, and Lemma 4 shows that (iii) implies (iv). Now, assume that (iv) holds. Let (X,U) be the principal solution and $(\tilde{X},\tilde{U})$ the associated solution of (S) at 0, so that

$$Q:=UX^\dagger+(UX^\dagger\tilde{X}-\tilde{U})(I-X^\dagger X)U^T$$

satisfies the assumptions of Lemma 3. Thus $P_k[Q]\geq 0$ and $R_k[Q]G_k=0$ hold; the latter statement because of Remark 1 (iv) and (v). Suppose now that (v) is true with some symmetric Q and pick an admissible

(x,u) with $x_0 = x_{N+1} = 0$. Then $R_k[Q]x_k = 0$ because of Remark 1 (iv) and Lemma 2 (i) yields

$$\mathcal{F}(x,u) = \sum_{k=0}^{N} s_k^T P_k[Q] s_k \geq 0.$$

To show positive definiteness, assume that $\mathcal{F}(x,u)$ vanishes. Then $P_k[Q]s_k = 0$ for all $k \in J$ and Lemma 2 (ii) shows that $x = 0$. Thus $\mathcal{F} > 0$ and statements (i) through (v) are equivalent.

Now we will show the equivalence of conditions (vi) through (x). To do so, we perform a transformation of variables as follows:

$$\begin{cases} \tilde{z}_\mu := \mathcal{J}(\mathcal{K}+\mathcal{K}^T) z_{N-\mu+1} \\ \quad = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} z_{N-\mu+1}, & 0 \leq \mu \leq N+1; \\ \tilde{S}_\mu := (\mathcal{K}+\mathcal{K}^T) S_{N-\mu}^T (\mathcal{K}+\mathcal{K}^T) \\ \quad = \begin{pmatrix} \mathcal{D}_{N-\mu}^T & \mathcal{B}_{N-\mu}^T \\ \mathcal{C}_{N-\mu}^T & \mathcal{A}_{N-\mu}^T \end{pmatrix}, & 0 \leq \mu \leq N. \end{cases}$$

The system ($\tilde{\mathrm{S}}$) can be rewritten in terms of the new variables as

$$(\hat{\mathrm{S}}) \qquad \begin{cases} \tilde{z}_{N-k+1} = \mathcal{J}(\mathcal{K}+\mathcal{K}^T) z_k = \mathcal{J}(\mathcal{K}+\mathcal{K}^T) S_k^{-1} z_{k+1} \\ \qquad\qquad = (\mathcal{K}+\mathcal{K}^T) S_k^T \mathcal{J}^T z_{k+1} \\ \qquad\qquad = (\mathcal{K}+\mathcal{K}^T) S_k^T (\mathcal{K}+\mathcal{K}^T) \mathcal{J} (\mathcal{K}+\mathcal{K}^T) z_{k+1} \\ \qquad\qquad = \tilde{S}_{N-k} \tilde{z}_{N-k}, \quad k \in J \end{cases}$$

or equivalently

$$(\hat{\mathrm{S}}) \qquad \tilde{z}_{k+1} = \tilde{S}_k \tilde{z}_k, \quad k \in J,$$

and ($\hat{\mathrm{S}}$) is a system of the form (S) since $\tilde{S}$ is symplectic. We may apply what we have already shown to this system ($\hat{\mathrm{S}}$) and this yields right away the equivalence of (vii), (viii), (ix), and (x). To see that (vi) is equivalent to those conditions also we note that

$$\begin{aligned}\mathcal{F}(\tilde{z}) &= \sum_{k=0}^{N} \tilde{z}_k^T \{\tilde{S}_k^T \mathcal{K} \tilde{S}_k - \mathcal{K}\} \tilde{z}_k \\ &= \sum_{k=0}^{N} z_{N-k+1}^T \{\mathcal{J} S_{N-k} \mathcal{K}^T S_{N-k}^T \mathcal{J}^T - \mathcal{J}\mathcal{K}^T\mathcal{J}^T\} z_{N-k+1} \\ &= \sum_{k=0}^{N} z_{k+1}^T \{\mathcal{K} - S_k^{T-1} \mathcal{K} S_k^{-1}\} z_{k+1} = -\tilde{\mathcal{F}}(z)\end{aligned}$$

and that $\mathcal{K}z \neq 0$, $\mathcal{K}z_0 = \mathcal{K}z_{N+1} = 0$, $\mathcal{K}z = \mathcal{K}S^{-1}Ez$ if and only if $\mathcal{K}\tilde{z} \neq 0$, $\mathcal{K}\tilde{z}_0 = \mathcal{K}\tilde{z}_{N+1} = 0$, $\mathcal{K}E\tilde{z} = \mathcal{K}\tilde{S}\tilde{z}$ (use the identities $(\mathcal{K}+\mathcal{K}^T)^2 = \mathcal{J}^T\mathcal{J} = I$, $(\mathcal{K}+\mathcal{K}^T)\mathcal{K}(\mathcal{K}+\mathcal{K}^T) = \mathcal{K}^T$, $\mathcal{J}\mathcal{K}^T\mathcal{J}^T = -\mathcal{K}$, and $\mathcal{K}\mathcal{J}(\mathcal{K}+\mathcal{K}^T) = \mathcal{K}$). This shows the equivalence of conditions (vi) through (x).

To end the proof of our Reid roundabout theorem we now show that (i) and (vi) are equivalent. Assume that (i) holds and let z be such that

$$\mathcal{K}z \neq 0, \qquad \mathcal{K}z_0 = \mathcal{K}z_{N+1} = 0, \qquad \mathcal{K}z_k = \mathcal{K}S_k^{-1} z_{k+1}, \quad k \in J.$$

Put $z_k^* := S_k^{-1} z_{k+1}$ for $k \in J$ and $z_{N+1}^* := 0$. Then $\mathcal{K}z_k^* = \mathcal{K}S_k^{-1} z_{k+1} = \mathcal{K}z_k$ for all $k \in J$ and $\mathcal{K}z_{N+1}^* = 0 = \mathcal{K}z_{N+1}$. It follows that $\mathcal{K}z_{k+1}^* = \mathcal{K}z_{k+1} = \mathcal{K}S_k z_k^*$ holds for all $k \in J$ and (i) implies $0 < \mathcal{F}(z^*) = -\tilde{\mathcal{F}}(z)$ which shows that (vi) holds. Similarly, if (vi) is true, let z be such that

$$\mathcal{K}z \neq 0, \qquad \mathcal{K}z_0 = \mathcal{K}z_{N+1} = 0, \qquad \mathcal{K}z_{k+1} = \mathcal{K}S_k z_k, \quad k \in J$$

and put $z_{k+1}^* := S_k z_k$ for $k \in J$ and $z_0^* := 0$. Again it follows by (vi) that $0 > \tilde{\mathcal{F}}(z^*) = -\mathcal{F}(z)$ so that (i) holds, and this completes the proof. $\square$

Remark 5. We will now look at the reciprocal symplectic system

$$z_k^* = {S_k^*}^{-1} z_{k+1}^*, \quad k \in J \tag{S*}$$

with $z^* := \mathcal{J}^T z$ and

$$S^* := S^{T-1} = \mathcal{J} S \mathcal{J}^T = \begin{pmatrix} \mathcal{D} & -\mathcal{C} \\ -\mathcal{B} & \mathcal{A} \end{pmatrix}.$$

Again note that z^* (or Z^*) solves (S*) if and only if z (or Z) solves (S). Therefore we should strictly speak about focal points with respect to the system (S*) or with respect to the second component of a solution z of (S), etc., in the next theorem below, but for convenience we will use the same terminology as before.

Theorem 2 (Reid roundabout theorem for reciprocal symplectic systems). *The following statements are equivalent.*

(i) $\mathcal{F}(z) < 0$ *for all* z *with*

$$\mathcal{K}^T z \neq 0, \qquad \mathcal{K}^T z_0 = \mathcal{K}^T z_{N+1} = 0,$$

and

$$\mathcal{K}^T z_{k+1} = \mathcal{K}^T S_k z_k, \quad k \in J.$$

(ii) (S) *is disconjugate on* J*, i.e., no solution of* (S) *has more than one and no solution* z *of* (S) *with* $\mathcal{K}^T z_0 = 0$ *has any generalized zeros in* $(0, N+1]$*, where* $(m, m+1]$ *contains a generalized zero of a solution* z *of* (S) *if*

$$u_m \neq 0, \qquad u_{m+1} \in \operatorname{Im} \mathcal{C}_m, \quad \text{and} \quad u_m^T \mathcal{C}_m^\dagger u_{m+1} \geq 0 \quad \text{hold.}$$

(iii) *No solution* z *of* (S) *with* $\mathcal{K}^T z_0 = 0$ *has any generalized zero in* $(0, N+1]$.

(iv) *The solution* Z *of* (S) *with* $Z_0 = \begin{pmatrix} I \\ 0 \end{pmatrix}$ *has no focal points in* $(0, N+1]$*:*

$$\operatorname{Ker} U_{k+1} \subset \operatorname{Ker} U_k \quad \textit{and} \quad U_k U_{k+1}^\dagger \mathcal{C}_k \leq 0 \quad \textit{hold for all } k \in J.$$

(v) $R_k[Q]G_k = 0$ *has a symmetric solution* Q *on* J *with* $P_k[Q] \geq 0$; *where*

$$R_k[Q] = \begin{pmatrix} I \\ Q_{k+1} \end{pmatrix}^T S_k \mathcal{J}^T \begin{pmatrix} I \\ Q_k \end{pmatrix},$$
$$P_k[Q] = -\mathcal{C}_k^T \mathcal{A}_k - \mathcal{C}_k^T Q_{k+1} \mathcal{C}_k,$$

and

$$G_k = (\mathcal{D}_{k-1}\mathcal{D}_{k-2}\cdots\mathcal{D}_1\mathcal{C}_0 \quad \mathcal{D}_{k-1}\mathcal{D}_{k-2}\cdots\mathcal{D}_2\mathcal{C}_1 \quad \ldots \quad \mathcal{D}_{k-1}\mathcal{C}_{k-2} \quad \mathcal{C}_{k-1}).$$

(vi) $\tilde{\mathcal{F}}(z) > 0$ *for all* z *with*

$$\mathcal{K}^T z \neq 0, \qquad \mathcal{K}^T z_0 = \mathcal{K}^T z_{N+1} = 0,$$

and

$$\mathcal{K}^T z_k = \mathcal{K}^T S_k^{-1} z_{k+1}, \quad k \in J.$$

(vii) ($\tilde{\mathrm{S}}$) *is disconjugate on* J, *i.e., no solution of* (S) *has more than one and no solution* z *of* (S) *with* $\mathcal{K}^T z_{N+1} = 0$ *has any generalized zeros on* $[0, N+1)$, *where* $[m, m+1)$ *contains a generalized zero of a solution* z *of* (S) *if*

$$u_{m+1} \neq 0, \qquad u_m \in \operatorname{Im} \mathcal{C}_m^T, \quad \textit{and} \quad u_m^T \mathcal{C}_m^\dagger u_{m+1} \geq 0 \quad \textit{hold}.$$

(viii) *No solution* z *of* (S) *with* $\mathcal{K}^T z_{N+1} = 0$ *has a generalized zero on* $[0, N+1)$.

(ix) *The solution* Z *of* (S) *with* $Z_{N+1} = \left(\begin{smallmatrix} -I \\ 0 \end{smallmatrix}\right)$ *has no focal points in* $[0, N+1)$:

$$\operatorname{Ker} U_k \subset \operatorname{Ker} U_{k+1} \quad \textit{and} \quad U_{k+1} U_k^\dagger \mathcal{C}_k^T \leq 0 \quad \textit{hold for all } k \in J.$$

(x) $\tilde{R}_k[Q]\tilde{G}_k = 0$ *has a symmetric solution* Q *on* J *with* $\tilde{P}_k[Q] \geq 0$; *where*

$$\tilde{R}_k[Q] = \begin{pmatrix} Q_k \\ I \end{pmatrix}^T S_k^T \mathcal{J} \begin{pmatrix} Q_{k+1} \\ I \end{pmatrix},$$
$$\tilde{P}_k[Q] = -\mathcal{C}_k \mathcal{D}_k^T - \mathcal{C}_k Q_k \mathcal{C}_k^T,$$

and

$$\tilde{G}_k = (\mathcal{A}_{k+1}^T \mathcal{A}_{k+2}^T \cdots \mathcal{A}_{N-1}^T \mathcal{C}_N^T \quad \mathcal{A}_{k+1}^T \mathcal{A}_{k+2}^T \cdots \mathcal{A}_{N-2}^T \mathcal{C}_{N-1}^T \quad \dots \quad \mathcal{A}_{k+1}^T \mathcal{C}_{k+2}^T \quad \mathcal{C}_{k+1}^T).$$

Proof. This is just a restatement of items (vi), (vii), (ix), and (x) of Theorem 1 applied to the system (S*). □

We finish this section with two results on transformations of symplectic systems that will be needed later on. First, we give transformations that transform symplectic systems into other symplectic systems. We note that the proof of this result is much easier than the proof of the corresponding result for linear Hamiltonian difference systems which is given in [**18**, Theorem 1]. Secondly, we present transformations that preserve the important property of disconjugacy. Again, this result contains the more special result of [**18**, Corollary 3.1] for linear Hamiltonian difference system with nonsingular B.

Lemma 6. *Let* $\mathcal{R}$ *be symplectic. Then the transformation* $\tilde{z} := \mathcal{R}^{-1} z$ *takes the symplectic system* $Ez = Sz$ *into another symplectic system* $E\tilde{z} = \tilde{S}\tilde{z}$.

Proof. We have $\tilde{S}_k = \mathcal{R}_{k+1}^{-1} S_k \mathcal{R}_k$ because of

$$\tilde{z}_{k+1} = \mathcal{R}_{k+1}^{-1} z_{k+1} = \mathcal{R}_{k+1}^{-1} S_k z_k = \mathcal{R}_{k+1}^{-1} S_k \mathcal{R}_k \tilde{z}_k;$$

and the computation

$$\tilde{S}_k^T \mathcal{J} \tilde{S}_k = \mathcal{R}_k^T S_k^T \mathcal{R}_{k+1}^{T-1} \mathcal{J} \mathcal{R}_{k+1}^{-1} S_k \mathcal{R}_k = \mathcal{R}_k^T S_k^T \mathcal{J} S_k \mathcal{R}_k = \mathcal{R}_k^T \mathcal{J} \mathcal{R}_k = \mathcal{J}$$

shows that we have obtained another symplectic system using this transformation. □

Lemma 7. *Let H_k and K_k be $n \times n$-matrices with $H_0 = I$ such that the matrix*

$$\mathcal{R}_k = \begin{pmatrix} H_k & 0 \\ K_k & H_k^{T-1} \end{pmatrix}$$

is symplectic and put $z_k = \mathcal{R}_k \tilde{z}_k$. Then system (S) *is disconjugate on J if and only if the transformed system $\tilde{z}_{k+1} = \tilde{S}_k \tilde{z}_k$ is disconjugate on J, where, as in Lemma 6 above, $\tilde{S}_k = \mathcal{R}_{k+1}^{-1} S_k \mathcal{R}_k$.*

Proof. Let (X, U), $(\tilde{X}, \tilde{U})$ be the principal solution of (S) and the transformed system, respectively. Then $\tilde{X}_k = H_k^{-1} X_k$ and $\tilde{\mathcal{B}}_k = H_{k+1}^{-1} \mathcal{B}_k H_k^{T-1}$. Obviously we have $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ if and only if $\operatorname{Ker} \tilde{X}_{k+1} \subset \operatorname{Ker} \tilde{X}_k$ and (use Remark 1 (v))

$$\begin{aligned} \tilde{P}_k &= \tilde{X}_k \tilde{X}_{k+1}^{\dagger} \tilde{\mathcal{B}}_k = H_k^{-1} X_k (H_{k+1}^{-1} X_{k+1})^{\dagger} H_{k+1}^{-1} \mathcal{B}_k H_k^{T-1} \\ &= H_k^{-1} X_k X_{k+1}^{+} H_{k+1} H_{k+1}^{-1} X_{k+1} (H_{k+1}^{-1} X_{k+1})^{\dagger} \\ &\quad \times H_{k+1}^{-1} X_{k+1} X_{k+1}^{\dagger} \mathcal{B}_k H_k^{T-1} \\ &= H_k^{-1} X_k X_{k+1}^{\dagger} \mathcal{B}_k H_k^{T-1} = H_k^{-1} P_k H_k^{T-1}. \end{aligned}$$

Thus $P_k \geq 0$ if and only if $\tilde{P}_k \geq 0$ what we needed to prove. $\square$

4. Reciprocity of linear Hamiltonian difference systems. In the theory of continuous linear Hamiltonian systems the so-called reciprocity principle plays an important role. It says that if the matrices B and C are nonnegative definite and both systems

$$x' = A(t)x + B(t)u, \qquad u' = -C(t)x - A^T(t)u \tag{1}$$

$$y' = -A^T(t)y + C(t)z, \qquad z' = -B(t)y + A(t)z \tag{2}$$

are identically normal for large t (i.e., if $x(t) \equiv 0$ ($u(t) \equiv 0$) on a nondegenerate subinterval of an interval $[T, \infty)$, T sufficiently large, then $(x, u) \equiv (0, 0)$—an alternative terminology is controllable for large t—see [**14**]) then (1) is nonoscillatory at ∞ if and only if (2) is nonoscillatory at ∞, see [**35**].

This reciprocity principle is of particular importance when (1) and (2) correspond to self-adjoint, even order, differential equations

$$(-1)^n (r(t) y^{(n)})^{(n)} = w(t) y \tag{3}$$

and

$$(-1)^n\left(\frac{1}{w(t)}y^{(n)}\right)^{(n)} = \frac{1}{r(t)}y, \tag{4}$$

where $r(t), w(t) > 0$. By this principle, equation (3) is nonoscillatory at ∞ if and only if (4) is nonoscillatory at ∞, and this statement has many applications in the spectral theory of singular differential operators, see [**5, 15, 28**].

The reciprocal symplectic system (S*) was introduced in Remark 5 as a symplectic system which results from (S) upon the transformation

$$\binom{y}{z} = \mathcal{J}^T\binom{x}{u}.$$

If (S) is a linear Hamiltonian difference system then $y = -u$, $z = x$ and (S*) may be written in the form

$$\Delta y_k = -A_k^T y_k + C_k z_{k+1}, \qquad \Delta z_k = -B_k y_k + A_k z_{k+1}. \tag{$\tilde{\mathrm{H}}$}$$

Substituting for $S_k = S_k^{(H)}$, compare Remark 2 (ii), the equation of motion for the quadratic functional $\tilde{\mathcal{F}}$ introduced in Theorem 2 reads

$$\Delta y_k = -A_k^T y_k + C_k z_{k+1}, \quad \text{i.e.,} \qquad \Delta u_k = -C_k x_{k+1} - A_k^T u_k$$

and the quadratic functional $\tilde{\mathcal{F}}(x,u)$ is

$$\begin{aligned}\tilde{\mathcal{F}}(x,u) = -\sum_{k=0}^{N}[-x_{k+1}^T(I - A_k^T - C_k\tilde{A}_kB_k)\tilde{A}_k^TC_kx_{k+1} \\ + 2u_{k+1}^T\tilde{A}_kB_k\tilde{A}_k^TC_kx_{k+1} + u_{k+1}^T\tilde{A}_kB_k\tilde{A}_k^Tu_{k+1}] \\ = \sum_{k=0}^{N}[x_{k+1}^TC_kx_{k+1} - u_k^TB_ku_k].\end{aligned}$$

In this section we prove that under certain additional assumptions system (H) is eventually disconjugate if and only if ($\tilde{\mathrm{H}}$) is eventually disconjugate. First we reformulate main definitions and statements of Sections 2 and 3 to linear Hamiltonian difference system (H) and its reciprocal system ($\tilde{H}$).

The concept of identical normality (controllability) of continuous LHS has the following discrete analogy introduced in [**9**, Definition 3].

Definition 5. System (H) is said to be *controllable for large* k if there exists an integer κ, the so-called *controllability index*, and $M \in \mathbf{N}$ such that $x_m = \dots = x_{m+\kappa} = 0$ for some $m \geq M$ implies $x_k = u_k \equiv 0$ on $[M, \infty) \cap \mathbf{Z}$.

Similarly, system ($\tilde{\mathrm{H}}$) is said to be controllable if $y_m = \dots = y_{m+\kappa} = 0$ implies $y_k = z_k \equiv 0$ on $[M, \infty) \cap \mathbf{Z}$.

In the next theorem we specify some items of Theorem 1 to linear Hamiltonian difference system and its reciprocal system. We restrict essentially our attention only on those parts which we use for investigation of eventual disconjugacy of (H) and ($\tilde{\mathrm{H}}$). Recall that by Definition 4 and Remark 3 applied to (H), a solution (x, u) of (H) has a *generalized zero* in $(m, m+1]$ if $x_m \neq 0$, $x_{m+1} = \tilde{A}_m B_m c$ for some $c \in \mathbf{R}^n$ and $x_m^T c \leq 0$. Similarly, by Theorem 2 (vii), a solution (y, z) of ($\tilde{\mathrm{H}}$) has a *generalized zero* in $[m, m+1)$ if $y_{m+1} \neq 0$, $y_m = \tilde{A}_m^T C_m c$ for some $c \in \mathbf{R}^n$ and $c^T y_{m+1} \leq 0$. Systems (H) and ($\tilde{\mathrm{H}}$) are said to be *eventually disconjugate* if there exists $N \in \mathbf{N}$ such that these systems are disconjugate on $[N, M]$ for every $M > N$, whereby disconjugacy on $[N, M]$ is defined in the same way as disconjugacy of (S) and (S*) on J introduced in Section 3.

Denote by $\mathcal{D}(N)$ the class of pairs of n-vector sequences (x, u) such that $\Delta x_k = A_k x_{k+1} + B_k u_k$ for $k \geq N$, $x_k \equiv 0$ for $k \leq N$ and only finitely many x_k are nonzero. Similarly, $\tilde{\mathcal{D}}(N)$ denotes pairs (x, u) such that $\Delta u_k = -C_k x_{k+1} - A_k^T u_k$, only finitely many $u_k \neq 0$ for $k \geq N$ and $u_k \equiv 0$ for $k \leq N$.

Proposition 1. *Suppose that* (H) *is controllable for large* k. *Then the following statements are equivalent.*

(i) *There exists* $N \in \mathbf{N}$ *such that*

$$\mathcal{F}(x, u) = \sum_{k=N}^{\infty} [u_k^T B_k u_k - x_{k+1}^T C_k x_{k+1}] > 0 \quad \textit{over } \mathcal{D}(N).$$

(ii) *System* (H) *is eventually disconjugate.*

(iii) *Let* (X, U) *be the solution of* (H) *with* $(X_N, U_N) = (0, I)$ *for some* $N \in \mathbf{N}$ *sufficiently large. Then* $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$*,* X *is eventually nonsingular and* $X_k X_{k+1}^{-1} \tilde{A}_k B_k \geq 0$ *for large* k*.*

(iv) *There exist* $N \in \mathbf{N}$ *and symmetric* $n \times n$*-matrices* Q_k *such that* $(I + B_k Q_k)^{-1} B_k \geq 0$ *for* $k \geq N$ *and*

$$Q_{k+1} = -C_k + \tilde{A}_k^{T-1} Q_k (I + B_k Q_k)^{-1} \tilde{A}_k^{-1}. \tag{5}$$

Proof. The statements (i), (ii) and (iii) are only immediate reformulations of the corresponding statements of Theorem 1 (in (iii) the eventual nonsingularity of X follows from controllability of (H)). Concerning the statement (iv), the eventual nonsingularity of X from (iii) implies that for $Q_k = U_k X_k^{-1}$ we have

$$\begin{aligned} R_k[Q] &= Q_{k+1} \tilde{A}_k (I + B_k Q_k) - [-C_k \tilde{A}_k + (-C_k \tilde{A}_k B_k + \tilde{A}_k^{T-1}) Q_k] \\ &= (Q_{k+1} + C_k) \tilde{A}_k (I + B_k Q_k) - \tilde{A}_k^{T-1} Q_k \\ &= [Q_{k+1} + C_k - \tilde{A}_k^{T-1} Q_k (I + B_k Q_k)^{-1} \tilde{A}_k^{-1}] \tilde{A}_k (I + B_k Q_k), \end{aligned}$$

hence $R_k[Q] = 0$ if and only if (5) holds.

Further, if $R_k[Q] = 0$

$$\begin{aligned} P_k[Q] &= B_k \tilde{A}_k^T (-C_k \tilde{A}_k B_k + \tilde{A}_k^{T-1} - Q_{k+1} \tilde{A}_k B_k) \\ &= B_k - B_k \tilde{A}_k^T (C_k + Q_{k+1}) \tilde{A}_k B_k \\ &= B_k - B_k Q_k (I + B_k Q_k)^{-1} B_k \\ &= (I + B_k Q_k - B_k Q_k)(I + B_k Q_k)^{-1} B_k \\ &= (I + B_k Q_k)^{-1} B_k. \end{aligned}$$

Controllability of (H) implies that the controllability matrices G given in Remark 1 (iv) (with indices k and 0 in the righthand side replaced by $N + k$ and N, respectively, and $\mathcal{A} = \tilde{A}, \mathcal{B} = \tilde{A}B$) eventually have rank n, hence $R_k[Q] G_k = 0$ if and only if $R_k[Q] = 0$ for large k. □

In a similar way, combining Theorems 1 and 2, we have

Proposition 2. *Suppose that system* ($\tilde{\mathrm{H}}$) *is controllable for large* k*. Then the following statements are equivalent.*

(i) *There exists* $N \in \mathbf{N}$ *such that*

$$\mathcal{F}(x,u) = \sum_{k=N}^{\infty} [u_k^T B_k u_k - x_{k+1}^T C_k x_{k+1}] < 0 \quad \textit{over } \tilde{\mathcal{D}}(N).$$

(ii) *System* $(\tilde{\mathrm{H}})$ *is eventually disconjugate.*

(iii) *Let* (Y,Z) *be the solution of* $(\tilde{\mathrm{H}})$ *with* $(Y_N, Z_N) = (0, I)$ *for some* $N \in \mathbf{N}$. *Then* $\operatorname{Ker} Y_{k+1} \subset \operatorname{Ker} Y_k$, Y *is eventually nonsingular and* $Y_{k+1} Y_k^{-1} \tilde{A}_k^T C_k \geq 0$ *for large* k.

(iv) *There exist* $N \in \mathbf{N}$ *and symmetric* $n \times n$*-matrices* V_k *such that* $(I - C_k V_{k+1})^{-1} C_k \geq 0$ *for* $k \geq N$ *and*

$$V_k = B_k + \tilde{A}_k^{-1} V_{k+1} (I - C_k V_{k+1})^{-1} \tilde{A}_k^{T-1}. \tag{6}$$

In our considerations so-called *recessive solutions* of (H) and $(\tilde{\mathrm{H}})$ at ∞ play an important role. A conjoined basis (X,U) of (H) is said to be *recessive* at ∞ if X is eventually nonsingular and there exists another conjoined basis $(\tilde{X}, \tilde{U})$ with $\tilde{X}$ eventually nonsingular such that $X_k^T \tilde{U}_k - U_k^T \tilde{X}_k$ is nonsingular and $\lim_{k\to\infty} \tilde{X}_k^{-1} X_k = 0$. Recessive solution at ∞ of $(\tilde{\mathrm{H}})$ is defined in a similar way. We briefly show that the construction of a recessive solution of eventually disconjugate three recurrence equation given in [**2**, Theorem 4.1] applies also to eventually disconjugate general linear Hamiltonian difference systems which are controllable for large k.

Let $m \in \mathbf{N}$ be such that (H) is disconjugate and controllable on $[m, \infty)$ and (X,U) be the solution of (H) given by the initial condition $(X_m, U_m) = (0, I)$. Controllability and disconjugacy imply that X_k is nonsingular for $k \geq m + \kappa$, where κ is the controllability index, and disconjugacy implies that $P_k = (I + B_k Q_k)^{-1} B_k = X_k (X_k + B_k U_k)^{-1} B_k = X_k X_{k+1}^{-1} \tilde{A}_k B_k = X_k [X_{k+1}^{-1} \tilde{A}_k B_k X_k^{T-1}] X_k^T$ is nonnegative definite, hence also $\bar{B}_k = X_{k+1}^{-1} \tilde{A}_k B_k X_k^{T-1}$ is nonnegative definite. Let $N \geq m + \kappa$ and consider the solution of (H)

$$\bar{X}_k = X_k \left(\sum_{j=N}^{k-1} \bar{B}_j \right), \qquad \bar{U}_k = U_k \left(\sum_{j=N}^{k-1} \bar{B}_j \right) + X_k^{T-1}$$

(the fact that $(\bar{X}, \bar{U})$ is really a solution of (H) is proved, e.g., in [**20**, Proposition 2.2]). Since $\bar{X}_N = 0$ we have that $\bar{X}_k$ and hence $\sum_{j=N}^{k-1} \bar{B}_j$ is invertible for all $k \geq N + \kappa$, again due to controllability and disconjugacy of (H) on $[m, \infty)$. Then $X^T\bar{U} - U^T\bar{X} = I$ and $\bar{X}_k^{-1}X_k = (\sum_{j=N}^{k-1} \bar{B}_j)^{-1}$ for $k \geq N + \kappa$. Consequently, the solution (X, U) of (H) is recessive at ∞ if and only if

$$\lim_{k\to\infty} \left(\sum_{j=N}^{k} X_{j+1}^{-1}\tilde{A}_j B_j X_j^{T-1} \right)^{-1} = 0.$$

Similarly, the solution (Y, Z) of $(\tilde{\mathrm{H}})$ is recessive at ∞ if and only if

$$\lim_{k\to\infty} \left(\sum_{j=N}^{k} Y_j^{-1}\tilde{A}_j^T C_j Y_{j+1}^{T-1} \right)^{-1} = 0.$$

The following construction of the recessive solution at ∞ of eventually disconjugate and controllable system (H) is the same as in Ahlbrandt [**2**, p. 1601]. Denote $S_{k,N}(X,U) = \sum_{j=N}^{k-1} X_{j+1}^{-1}\tilde{A}_j B_j X_j^{T-1}$ and consider the solution of (H)

$$\tilde{X}_k = X_k[I + S_{k,N}(X,U)], \qquad \tilde{U}_k = U_k[I + S_{k,N}(X,U)] + X_k^{T-1}.$$

By a direct computation we have

$$X_k = \tilde{X}_k[I - S_{k,N}(\tilde{X},\tilde{U})], \qquad U_k = \tilde{U}_k[I - S_{k,N}(\tilde{X},\tilde{U})] - \tilde{X}_k^{T-1},$$

hence

$$I = [I - S_{k,N}(\tilde{X},\tilde{U})][I + S_{k,N}(X,U)].$$

Since the second factor in the last product is nondecreasing with k, the first factor is nonincreasing and $0 < S_{k,N}(\tilde{X},\tilde{U}) < I$ for all $k \geq N + \kappa$. Hence there exists a nonnegative definite limit $S_{\infty,N}(\tilde{X},\tilde{U}) = \lim_{k\to\infty} S_{k,N}(\tilde{X},\tilde{U})$. Now, it is easy to see that

$$\begin{aligned}
\hat{X}_k &= \tilde{X}_k[S_{\infty,N}(\tilde{X},\tilde{U}) - S_{k,N}(\tilde{X},\tilde{U})] \\
&= \tilde{X}_k \sum_{j=k}^{\infty} \tilde{X}_{j+1}^{-1}\tilde{A}_j B_j \tilde{X}_j^{T-1}, \\
\hat{U}_k &= \tilde{U}_k \sum_{j=k}^{\infty} \tilde{X}_{j+1}^{-1}\tilde{A}_k B_j \tilde{X}_j^{T-1} - \tilde{X}_k^{T-1}.
\end{aligned}$$

is a recessive solution of (H) at ∞.

The construction of the recessive solution of ($\tilde{\mathrm{H}}$) at ∞ is quite analogical. Recall only that recessive solution of ($\tilde{\mathrm{H}}$) at ∞ has essentially the same properties as the recessive solution of (H) at $-\infty$. For a more detailed study of this problem in case of the three recurrence matrix equation we refer to [**1, 2**].

Let (X, U) be the recessive solution of (H) at ∞. The solution $Q^- = UX^{-1}$ of the associated Riccati matrix equation $R[Q] = 0$ is said to be *distinguished* (another terminology is *eventually minimal*) at ∞. Similarly as in [**1**, Theorem 5.1] it may be shown that any solution Q of (5) which exists up to ∞ eventually satisfies the inequality $Q \geq Q^-$. For reciprocal system ($\tilde{\mathrm{H}}$), if (Y, Z) is the recessive solution at ∞, the associated solution $V^+ = ZY^{-1}$ of the time-reversed Riccati equation (6) is eventually maximal in the sense that any solution V of (6) which exists up to ∞ satisfies eventually the inequality $V \leq V^+$.

The transformation (see Lemma 7)

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & H^{T-1} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix},$$

where $H_0 = I$, $H_{k+1} = \tilde{A}_k H_k$, transforms (H) into the linear Hamiltonian difference system of the same form with $A_k = 0$. Since this transformation preserves oscillation behavior and controllability both of (H) and ($\tilde{\mathrm{H}}$), we may suppose without loss of generality that $A_k = 0$, i.e., $\tilde{A}_k = I$, in (H) and ($\tilde{\mathrm{H}}$), such systems we will call *reduced* Hamiltonian systems. Hence, in the remaining part of this section we deal with the reduced systems

$$(\mathrm{H_R}) \qquad \Delta x_k = B_k u_k, \qquad \Delta u_k = -C_k x_{k+1}$$

and

$$(\tilde{\mathrm{H}}_\mathrm{R}) \qquad \Delta y_k = C_k z_{k+1}, \qquad \Delta z_k = -B_k y_k.$$

Observe that in case $\tilde{A}_k \equiv I$ we may write the Riccati operator in the form

$$(7) \qquad R_k[Q] = \Delta Q_k + C_k + Q_k(I + B_k Q_k)^{-1} B_k Q_k.$$

Indeed, we have

$$\begin{aligned} Q_{k+1} + C_k - Q_k(I + B_kQ_k)^{-1} &= \Delta Q_k + C_k - Q_k(I + B_kQ_k)^{-1} + Q_k \\ &= \Delta Q_k + C_k + Q_k(I + B_kQ_k)^{-1}B_kQ_k. \end{aligned}$$

Similarly, for the time-reversed equation (6) we have in case $\tilde{A}_k \equiv I$

$$\Delta V_k = -B_k - V_{k+1}(I - C_kV_{k+1})^{-1}C_kV_{k+1}. \tag{8}$$

Lemma 8. *Suppose that $C_k \geq 0$ for large k, Q is a symmetric solution of (5) on $[M, \infty)$, $M \in \mathbf{N}$, such that $(I + B_kQ_k)^{-1}B_k \geq 0$ for $k \geq M$. If $\tilde{Q}$ is any symmetric solution of*

$$\Delta Q_k = -Q_k(I + B_kQ_k)^{-1}B_kQ_k \tag{9}$$

such that $\tilde{Q}_M \geq Q_M$, then $\tilde{Q}$ exists on the whole interval $[M, \infty)$ and satisfies there inequalities $(I + B_k\tilde{Q}_k)^{-1}B_k \geq 0$, $\tilde{Q}_k \geq Q_k$.

Proof. Let (X, U) be the solution of $(\mathrm{H_R})$ given by the initial condition $X_M = I, U_M = Q_M$. Then $Q_k = U_kX_k^{-1}$, X_k is nonsingular and $X_kX_{k+1}^{-1}\tilde{A}_kB_k = (I + B_kQ_k)^{-1}B_k \geq 0$ for $k \geq M$, i.e., (X, U) has no focal point in $[M, \infty)$. By [**11**, Theorem 3] this is equivalent to the fact that for any $N > M$

$$x_M^TQ_Mx_M + \sum_{k=M}^{N}[u_k^TB_ku_k - x_{k+1}^TC_kx_{k+1}] > 0$$

for any (x, u) satisfying $\Delta x_k = B_ku_k$, $k \in [M, N] \cap \mathbf{Z}$, $x_{N+1} = 0$ and $x \not\equiv 0$. Now, let $(\tilde{X}, \tilde{U})$ be the solution of $\Delta X_k = B_kU_k$, $\Delta U_k = 0$ satisfying $\tilde{X}_M = I$, $\tilde{U}_M = \tilde{Q}_M$. Then for any $N > M$ and any nontrivial (x, u) satisfying $\Delta x_k = B_ku_k$, $k \in [M, N] \cap \mathbf{Z}$, $x_{N+1} = 0$ the last inequality implies

$$x_M^T\tilde{Q}_Mx_M + \sum_{k=M}^{N}u_k^TB_ku_k > 0,$$

hence by the above mentioned Theorem 3 of [**11**] $(\tilde{X}, \tilde{U})$ has no focal point in $[M, \infty)$, i.e., $\tilde{X}_k$ is nonsingular and $\tilde{X}_k\tilde{X}_{k+1}^{-1}\tilde{A}_kB_k = (I + B_k\tilde{Q}_k)^{-1}B_k \geq 0$ for $k \geq M$.

To prove the inequality $Q_k \geq \tilde{Q}_k$, we proceed similarly as in [**22**, Theorem 2]. Consider the matrix functional

$$\mathcal{F}(X,U) = \sum_{k=M}^{N} [U_k^T B_k U_k - X_{k+1}^T C_k X_{k+1}].$$

Substituting $(X,U) = (\tilde{X},\tilde{U})$ and using the Picone's identity we have

$$\begin{aligned}
\mathcal{F}(\tilde{X},\tilde{U}) &= \tilde{X}_k^T Q_k \tilde{X}_k \Big|_M^{N+1} \\
&\quad + \sum_{k=M}^{N} (\tilde{U}_k - Q_k\tilde{X}_k)^T (I + B_k Q_k)^{-1} B_k (\tilde{U}_k - Q_k \tilde{X}_k) \\
&\geq \tilde{X}_k^T Q_k \tilde{X}_k \Big|_M^{N+1}.
\end{aligned}$$

On the other hand, since $C_k \geq 0$, we have (again by the Picone's identity)

$$\begin{aligned}
\mathcal{F}(\tilde{X},\tilde{U}) &\leq \sum_{k=M}^{N} \tilde{U}_k^T B_k \tilde{U}_k \\
&= \tilde{X}_k^T \tilde{Q}_k \tilde{X}_k \Big|_M^{N+1} \\
&\quad + \sum_{k=M}^{N} (\tilde{U}_k - \tilde{Q}_k\tilde{X}_k)^T (I + B_k \tilde{Q}_k)^{-1} B_k (\tilde{U}_k - \tilde{Q}_k \tilde{X}_k) \\
&= \tilde{X}_k^T \tilde{Q}_k \tilde{X}_k \Big|_M^{N+1}.
\end{aligned}$$

Hence

$$\tilde{X}_k^T Q_k \tilde{X}_k \Big|_M^{N+1} \leq \tilde{X}_k^T \tilde{Q}_k \tilde{X}_k \Big|_M^{N+1}$$

and thus

$$\tilde{X}_{N+1}^T (Q_{N+1} - \tilde{Q}_{N+1}) \tilde{X}_{N+1} \leq \tilde{X}_M^T (Q_M - \tilde{Q}_M) \tilde{X}_M \leq 0.$$

Since $\tilde{X}_{N+1}$ is nonsingular, we have $Q_{N+1} \leq \tilde{Q}_{N+1}$ for any $N \geq M$. □

Lemma 9. *Suppose that* $(\mathrm{H_R})$ *is eventually disconjugate, controllable for large* k*, and let* Q^- *be the eventually minimal solution of the associated Riccati equation* $R[Q]=0$ *given by* (7)*. If* $C_k \geq 0$ *for large* k *and* $(\sum_{j=N}^k B_j)^{-1} \to 0$ *as* $k \to \infty$*, then* $Q_k^- \geq 0$ *for large* k*.*

Proof. Suppose that there exist $\alpha \in \mathbf{R}^n$ and $m \in \mathbf{N}$ arbitrarily large, such that $\alpha^T Q_m^- \alpha < 0$. Since the matrix sequence $\{Q_k^-\}$ is nonincreasing, $\alpha^T Q_k^- \alpha < 0$ for $k \geq m$. The assumption $(\sum_{j=N}^k B_j)^{-1} \to 0$ as $k \to \infty$ implies that $(X, U) = (I, 0)$ is the recessive solution at ∞ of the system

$$\Delta x_k = B_k u_k, \qquad \Delta u_k = 0,$$

hence $\tilde{Q}^- \equiv 0$ is the eventually minimal solution of the corresponding Riccati equation (9). Let W be the solution of (9) given by the initial condition $W_m = Q_m^-$. If m is sufficiently large, assumptions of Lemma 8 are satisfied, hence W exists up to ∞ and $W_k \geq Q_k^-$ for $k \geq m$. The matrix sequence $\{W_k\}$ is nonincreasing, thus $\alpha^T W_k \alpha < 0 \equiv \alpha^T \tilde{Q}_k^- \alpha$ for $k \geq m$ which contradicts the fact that $\tilde{Q}^- \equiv 0$ is the eventually minimal solution of (9). $\square$

Using essentially the same argument as in Lemmas 8 and 9 and using the reciprocal Picone's identity we may prove the following statement.

Lemma 10. *Suppose that* $(\tilde{\mathrm{H}}_\mathrm{R})$ *is eventually disconjugate, controllable for large* k*, and let* V^+ *be the eventually maximal solution at* ∞ *of the time-reversed Riccati equation* (8)*. If* $B_k \geq 0$ *for large* k *and* $(\sum_{j=N}^k C_j)^{-1} \to 0$ *as* $k \to \infty$ *then* $V_k^+ \leq 0$ *for large* k*.*

Now we are ready to prove the statement concerning relation between eventual disconjugacy of (H) and $(\tilde{\mathrm{H}})$. As we pointed out above, without loss of generality one may suppose that $A \equiv 0$, i.e., we consider the reduced systems $(\mathrm{H_R})$ and $(\tilde{\mathrm{H}}_\mathrm{R})$ instead of (H) and $(\tilde{\mathrm{H}})$.

Theorem 3. *Suppose that* $(\mathrm{H_R})$ *and its reciprocal system* $(\tilde{\mathrm{H}}_\mathrm{R})$ *are controllable for large* k (*not necessarily with the same controllability index*) *and* $B_k \geq 0$*,* $C_k \geq 0$ *for large* k*. Then* $(\mathrm{H_R})$ *is eventually disconjugate if and only if* $(\tilde{\mathrm{H}}_\mathrm{R})$ *is eventually disconjugate.*

Proof. Let $(\mathrm{H_R})$ be eventually disconjugate and Q^- be the eventually minimal solution at ∞ of the associated Riccati equation $R[Q] = 0$ given by (7). First suppose that $(\sum_{j=N}^k B_j)^{-1} \to 0$ as $k \to \infty$. By Lemma 9 $Q_k^- \geq 0$ for large k. We shall show that controllability of $(\tilde{\mathrm{H}})$ implies that actually $Q_k^- > 0$. The sequence Q_k^- is nonincreasing, hence there exists $m \in \mathbf{N}$ such that $\operatorname{rank} Q_k^-$ and $\operatorname{ind} Q_k^-$ are constant for $k \geq m$ ($\operatorname{ind} Q_k^-$ denotes the index of Q_k^-, i.e., the number of negative eigenvalues). This follows from monotonicity of Q_k^- since $\operatorname{ind} Q_k^-$ is nondecreasing and if $\operatorname{rank} Q_k^-$ changes its value, some eigenvalue, before being zero, becomes negative. This change of $\operatorname{rank} Q_k^-$ and $\operatorname{ind} Q_k^-$ may happen only finitely many times, hence there exists $m \in \mathbf{N}$ with the claimed property.

Now, by (7) we have

$$Q_{m+k+1}^- = Q_m - \sum_{j=m}^{m+k} C_j - \sum_{j=m}^{m+k} Q_j^-(I + B_j Q_j^-)^{-1} B_j Q_j^-,$$

hence for any $\alpha \in \mathbf{R}^n$

$$\alpha^T Q_{m+k+1}^- \alpha \leq \alpha^T Q_m^- \alpha - \alpha^T \left(\sum_{j=m}^{m+k} C_j \right) \alpha.$$

If $\alpha \neq 0$ and $k \geq m + \kappa$, κ being the controllability index of $(\tilde{\mathrm{H}})$, at least one of the terms $\alpha^T C_j \alpha$ must be positive, hence $Q_{m+k+1}^- < Q_k^-$, but this contradicts the fact that rank Q_k^- and $\operatorname{ind} Q_k^-$ do not change for $k \geq m$. Consequently Q_k^- is eventually nonsingular and hence, by Lemma 9, positive definite.

Set $V_k = -(Q_k^-)^{-1} = -X_k U_k^{-1}$. Directly one may verify that V_k is a solution of the time-reversed Riccati equation (8) and since $Q_k^- > 0$ this solution exists up to infinity. To prove eventual disconjugacy of $(\tilde{\mathrm{H}}_{\mathrm{R}})$ we need to show that $(I - C_k V_{k+1})^{-1} C_k \geq 0$ for large k. We have $U_{k+1} = U_k - C_k X_{k+1}$, hence

$$\begin{aligned} 0 \leq C_k &= U_k X_{k+1}^{-1} - U_{k+1} X_{k+1}^{-1} \\ &= X_{k+1}^{T-1} [X_{k+1}^T U_k - X_{k+1}^T U_{k+1}] X_{k+1}^{-1}, \end{aligned}$$

i.e., $X_{k+1}^T U_k \geq X_{k+1}^T U_{k+1} = X_{k+1}^T Q_{k+1}^- X_{k+1} > 0$, thus $U_k^{-1} X_{k+1}^{T-1} \leq U_{k+1}^{-1} X_{k+1}^{T-1}$ and

$$\begin{aligned}(I - C_k V_{k+1})^{-1} C_k &= U_{k+1} U_k^{-1} C_k = U_{k+1} X_{k+1}^{-1} - U_{k+1} U_k^{-1} U_{k+1} X_{k+1}^{-1} \\ &= U_{k+1}[U_{k+1}^{-1} X_{k+1}^{T-1} - U_k^{-1} X_{k+1}^{T-1}] U_{k+1}^T \geq 0.\end{aligned}$$

Now, if $(\sum_{j=N}^k B_j)^{-1} \not\to 0$ as $k \to \infty$, replace B by a matrix $\overline{B}$ for which $\overline{B} \geq B$ and $(\sum_{j=N}^k \overline{B}_j)^{-1} \to 0$ as $k \to \infty$. By the previous argument the system

$$\Delta x_k = \overline{B}_k u_k, \qquad \Delta u_k = -C_k x_{k+1}$$

is eventually disconjugate and by Proposition 2

$$\overline{\mathcal{F}}(x,u) = \sum_{k=N}^{\infty} [u_k^T \overline{B}_k u_k - x_{k+1}^T C_k x_{k+1}] < 0$$

for any nontrivial $(x,u) \in \tilde{\mathcal{D}}(N)$ where $N \in \mathbf{N}$ is sufficiently large. Since $B \leq \overline{B}$, the same holds for $\mathcal{F}$, i.e., $(\tilde{\mathrm{H}}_{\mathrm{R}})$ is also eventually disconjugate.

Conversely, suppose that $(\tilde{\mathrm{H}}_{\mathrm{R}})$ is eventually disconjugate and let V^+ be the eventually maximal solution at ∞ of the time-reversed Riccati equation (8). If $(\sum_{j=N}^k C_j)^{-1} \to 0$ as $k \to \infty$ then by Lemma 10 $V_k^+ \leq 0$ for large k and controllability of (H) implies in the same way as in the previous part of the proof that $V_k^+ < 0$ and hence $Q_k = -(V_k^+)^{-1}$ is a solution of (9) which exists up to ∞. Since $X_{k+1} = X_k + B_k U_k$, we have

$$0 \leq B_k = X_{k+1} U_k^{-1} - X_k U_k^{-1} = U_k^{T-1}[U_k^T X_{k+1} - U_k^T X_k] U_k^{-1},$$

i.e., $U_k^T X_{k+1} \geq U_k^T X_k = -U_k^T V_k^+ U_k > 0$ which implies $X_{k+1}^{-1} U_k^{T-1} \leq X_k^{-1} U_k^{T-1}$. Consequently,

$$\begin{aligned}P_k[Q] = (I + B_k Q_k)^{-1} B_k &= X_k X_{k+1}^{-1} B_k \\ &= X_k U_k^{-1} - X_k X_{k+1}^{-1} X_k U_k^{-1} \\ &= X_k [U_k^{-1} X_k^{T-1} - X_{k+1}^{-1} U_k^{T-1}] X_k^T \geq 0.\end{aligned}$$

This means that $(\mathrm{H_R})$ is eventually disconjugate. Finally, if $(\sum_{j=N}^{k} C_j)^{-1} \not\to 0$ as $k \to \infty$, replace C_k by $\overline{C}_k$ such that $\overline{C}_k \geq C_k$, $(\sum_{j=N}^{k} \overline{C}_j)^{-1} \to 0$ as $k \to \infty$ and apply the same argument as in the first part of the proof. □

Remark 6. (i) Consider the self-adjoint, even order, two-term difference equation (which is a special form of (SL))

$$(-1)^n \Delta^n (r_k \Delta^n y_k) = w_k y_{k+n}, \tag{10}$$

where $r_k, w_k > 0$. Directly one may verify that $z_k = r_k \Delta^n y_k$ solves the (reciprocal) equation

$$(-1)^n \Delta^n \left(\frac{1}{w_k} \Delta^n z_k \right) = \frac{1}{r_{k+n}} y_{k+n}. \tag{11}$$

Equations (10) and (11) may be written in the form of (H) and $(\tilde{\mathrm{H}})$ with A, B, C of the form given in Remark 2 (i) and

$$\begin{aligned} x_k &= \begin{pmatrix} y_{k+n-1} \\ \Delta y_{k+n-2} \\ \vdots \\ \Delta^{n-1} y_k \end{pmatrix}, \\ u_k &= \begin{pmatrix} (-1)^{n-1} \Delta^{n-1} (r_k \Delta^n y_k) \\ \vdots \\ -\Delta (r_k \Delta^n y_k) \\ r_k \Delta^n y_k \end{pmatrix} =: \begin{pmatrix} (-1)^{n-1} \Delta^{n-1} z_k \\ \vdots \\ -\Delta z_k \\ z_k \end{pmatrix}. \end{aligned} \tag{12}$$

Hartman [**24**] defined a generalized zero of order n for (10) with $r_k = 1$ as follows. A solution y of (10) is said to have a *generalized zero point of multiplicity* n *at* $k+1$ if $y_k \neq 0$, $y_{k+1} = \cdots = y_{k+n-1} = 0$ and $(-1)^n y_k y_{k+n} \geq 0$. In [**10**] it was shown that the definition of a generalized zero point in interval $(k, k+1]$ for linear Hamiltonian difference systems (H) with A, B, C given by Remark 2 (i) complies with Hartman's definition.

Similarly, if $[m, m+1)$ contains a generalized zero of $(\tilde{\mathrm{H}})$, i.e., there exists $c = (c_1, \ldots, c_n)^T \in \mathbf{R}^n$ such that $u_m = \tilde{A}^T C_m c$, $u_{m+1}^T c \leq 0$

then, taking into account that u_m is given by (12), we have $u_m = \tilde{A}^T C_m c$ if and only if $z_m = w_m c_1$, $z_{m+1} = \cdots = z_{m+n-1} = 0$ and $u_{m+1}^T c = (-1)^{n-1} \frac{1}{w_m} z_m z_{m+n}$. Here we have only one difference with respect to the Hartman's definition, the condition $z_m \neq 0$ is replaced by the condition $z_{m+n} \neq 0$, but this difference is immaterial. Since systems (H) and ($\tilde{\mathrm{H}}$) corresponding to (10) and (11) are controllable with controllability index n, see [**9**], Theorem 3 implies that (10) is eventually disconjugate if and only if (11) is eventually disconjugate in the sense that there exists $N \in \mathbf{N}$ such that the interval $[N, \infty)$ contains no pair of generalized zeros of order n.

(ii) Theorem 3 establishes duality between eventual disconjugacy of original and reciprocal system only for linear Hamiltonian difference systems (H) and its reciprocal ($\tilde{\mathrm{H}}$) since in the proof we needed the fact that the matrix $\tilde{A} = (I - A)^{-1}$ is nonsingular (which is equivalent to possibility to transform (H) into the system ($\mathrm{H_R}$) with $A \equiv 0$, i.e., $\tilde{A} \equiv 0$). We conjecture that under the assumption $\mathcal{A}^T \mathcal{C} \leq 0$, $\mathcal{B}\mathcal{A}^T \geq 0$ and suitable controllability assumption this statement also holds for a general symplectic system (S) and its reciprocal (S^*).

5. Transformations for Sturm-Liouville equations. In this section we establish a discrete version of Theorem 2.1 of [**15**]. In the continuous case this transformation turns out to be a useful tool in oscillation theory of Sturm-Liouville equations of higher order, cf., e.g., [**16**].

Theorem 4. *Let $h_k > 0$, $L(y) = \sum_{\nu=0}^{n} (-1)^\nu \Delta^\nu (r_k^{(\nu)} \Delta^\nu y_{k+n-\nu})$ and consider the transformation $y_k = h_k z_k$. Then we have*

$$h_{k+n} L(y) = \sum_{\nu=0}^{n} (-1)^\nu \Delta^\nu (R_k^{(\nu)} \Delta^\nu z_{k+n-\nu}),$$

where $R_k^{(n)} = h_{k+n} h_k r_k^{(n)}$ and $R_k^{(0)} = h_{k+n} L(h)$.

Proof. We proceed similarly as in the continuous case treated in [**15**]. We write equation $L(y) = 0$ in the form of linear Hamiltonian difference system (H) and we consider the transformation of (H)

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} H & 0 \\ K & H^{T-1} \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

We shall show that the $n \times n$-matrices H, K can be chosen in such a way that this transformation transforms (H) into the system

$$(\overline{\mathrm{H}}) \qquad \Delta y_k = \bar{A}_k y_{k+1} + \bar{B}_k z_k, \qquad \Delta z_k = -\bar{C}_k y_{k+1} - \bar{A}_k z_k$$

with $\overline{A} = A$, $\overline{B}_k = \operatorname{diag}\{0, \dots, 0, 1/(h_{n+k} h_k r_k^{(n)})\}$, and $\overline{C}_k$ diagonal.

According to Corollary 3.1 of [**18**]

$$\begin{aligned}
\overline{A}_k &= I - (H_k + B_k K_k)^{-1} \tilde{A}_k^{-1} H_{k+1}, \\
\overline{B}_k &= (H_k + B_k K_k)^{-1} B_k H_k^{T-1}, \\
\overline{C}_k &= -H_{k+1}^T [-K_{k+1} - C_k H_{k+1} \\
&\qquad + \tilde{A}_k^{T-1} K_k (H_k + B_k K_k)^{-1} \tilde{A}_k^{-1} H_{k+1}].
\end{aligned}$$

Let

$$H_k = H_k^{(i,j)} = \binom{i-1}{j-1} \Delta^{i-j} h_{k+n-i}, \quad i, j = 1, \dots, n. \tag{13}$$

The identity $\overline{A} = A$ gives

$$(H_k + B_k K_k)(I - A) = (I - A) H_{k+1} \tag{14}$$

and this identity determines the last row of K. Indeed, directly one may verify that the first $n-1$ rows of H given by (13) satisfy (14) and

$$\begin{aligned}
[(I - A) H_{k+1}]^{(n,j)} &= \binom{n-1}{j-1} \Delta^{n-j} h_{k+1}, \\
[(H + BK)(I - A)]_k^{n,j} &= \binom{n}{j-1} \Delta^{n-j} h_k \\
&\quad - \binom{n-1}{j-2} \Delta^{n-j} h_{k+1} \\
&\quad + \frac{1}{r_k^{(n)}} (K^{n,j} - K^{n,j-1})
\end{aligned}$$

(with the convention $K^{i,j} = 0$ if $i = 0$ or $j = 0$ and $\binom{n}{l} = 0$ if $l < 0$). Particularly, $K^{n,1} = r_k^{(n)} \Delta^n h_k$.

Next we prove that the matrix $H+BK$ is nonsingular and compute the matrix $\overline{B}$. From (14) we have $H_{k+1}^{n,j} = (H+BK)_k^{n,j} - (H+BK)_k^{n,j-1}$, hence

$$(H+BK)_k^{n,n} = \sum_{j=1}^{n} H_{k+1}^{n,j} = \sum_{j=1}^{n} \binom{n-1}{j-1} \Delta^{n-j} h_{k+1} = h_{k+n}.$$

Since the matrix $(H+BK)$ is lower triangular, $\det(H+BK)_k = \prod_{j=1}^{n} h_{k+j} > 0$.

We have $\overline{B} = (H+BK)^{-1} B H^{T-1}$, where B has the only nonzero entry $B^{n,n}$ and the matrix H_k^{T-1} is upper triangular, hence $\overline{B}$ is diagonal with only nonzero entry in the right lower corner where

$$\overline{B}_k^{n,n} = \left[(H+BK)^{-1} B H^{T-1}\right]_k^{n,n} = \frac{1}{h_{k+n}} \frac{1}{r_k^{(n)}} \frac{1}{h_k}$$

since

$$(H_k^{-1})^{i,j} = \binom{i-1}{j-1} \Delta^{i-j} \left(\frac{1}{h_{k+n-i}}\right).$$

Finally, we show that the remaining entries of K can be chosen such that the matrix $\overline{C}$ in ($\overline{\mathrm{H}}$) is diagonal. Using the identity $\overline{A} = A$ we have

$$\overline{C}_k = -H_{k+1}^T[-K_{k+1} - C_k H_{k+1} + (I-A^T) K_k (I-A)]. \tag{15}$$

First we determine $K^{i,j}$, $1 \le i \le n-1, 1 \le j \le i$, in such a way that the matrices $M_k := [-K_{k+1} + C_k H_{k+1} + (I-A^T) K_k (I-A)]$ have only zero entries below the diagonal. We have $M_k^{i,j} = -K_{k+1}^{i,j} - C_k^{i,i} H_{k+1}^{i,j} + K_k^{i,j} - K_k^{i-1,j} - K_k^{i,j-1} + K_k^{i-1,j-1}$, hence if

$$K_k^{i-1,j} = -K_{k+1}^{i,j} - C_k^{i,i} H_{k+1}^{i,j} + K_k^{i,j} - K_k^{i,j-1} + K_k^{i-1,j-1} \tag{16}$$

for $2 \le i \le n, 1 \le j < i$, M is upper triangular. Now let us write K in the form $K = \tilde{K} + \overline{K}$, where $\overline{K}$ is upper triangular with zeros on the diagonal and $\tilde{K}$ is lower triangular with entries determined by (16). The identity $H^T K = K^T H$ (which follows from the symplecticity of the transformation matrix converting (H) into ($\overline{\mathrm{H}}$))

yields $H^T\tilde{K} - \tilde{K}^T H = \overline{K}^T H - H^T\overline{K}$. The matrix $\tilde{K}^T H - H^T\tilde{K}$ is antisymmetric, hence the last identity may be written in the form $\overline{K}^T H - H^T\overline{K} = U - U^T$, where U is a lower triangular matrix with zeros on the diagonal. Setting $\overline{K} = H^{T-1}U^T$ we have defined all entries of K. Since both M_k and H^T_{k+1} are upper triangular, $\overline{C}_k$ has the same property and its symmetry implies that $\overline{C}_k$ is diagonal. It remains to show that $\overline{C}_k^{1,1} = -R_k^{(0)} = -h_{n+k}L(h)$. From (16) and the fact that $K_k^{n,1} = r_k^{(n)}\Delta^n h_k$ follows

$$\Delta K_k^{1,1} = K_{k+1}^{1,1} - K_k^{1,1} = \sum_{\nu=1}^{n}(-1)^{\nu-1}\Delta^\nu(r_k^{(\nu)}\Delta^\nu h_{k+n-\nu}),$$

and by (15)

$$\begin{aligned}\overline{C}_k^{1,1} &= -(H^T_{k+1})^{1,1}[-K_{k+1}^{1,1} + K_k^{1,1} - C_k^{1,1}H_{k+1}^{1,1}]\\ &= -h_{k+n}\bigg[\sum_{\nu=1}^{n}(-1)^{\nu}\Delta^\nu(r_k^{(\nu)}\Delta^\nu h_{k+n-\nu}) + r_k^{(0)}h_{k+n}\bigg]\\ &= -h_{k+n}L(h),\end{aligned}$$

Remark 7. In the previous theorem we computed only the first and last coefficients $R_k^{(n)}$ and $R_k^{(0)}$ of the transformed equation. The remaining coefficients can be also computed explicitly (in the continuous case this is done [**17**, Theorem 3.1]), but the formulae are rather complicated and in most applications only formulae for the first and last coefficients are needed.

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